D. Electron has an intrinsic Spin Angular Momentum

- Stern-Gerlach exp't: A new angular momentum due to electron
- Intrinsic property of electron: (Meaning - Every electron has this property)

  [c.f.: charge $(-e)$ of electron, mass m_e of electron]
- Goudsmit and Uhlenbeck (1925) [Pauli: "an additional quantum number"]

This is electron's "Spin"

$$\boxed{\begin{array}{l}\text{Every electron has a spin angular momentum } \vec{S} \\ \bullet\ \hat{S}^2 \text{ has only one eigenvalue } \frac{3}{4}\hbar^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \\ \bullet\ \hat{S}_z \text{ takes on only } +\frac{1}{2}\hbar \text{ and } -\frac{1}{2}\hbar\end{array}} \quad (9)$$

Putting it in standard Angular momentum form:

(10)

$\hat{S}^2$ has one eigenvalue[+] $s(s+1)\hbar^2$

with s always takes on ½ for an electron

With $s = 1/2$, $m_s = +1/2, -1/2$ where $m_s\hbar$ are eigenvalues of $\hat{S}_z$

This "$s = 1/2$ only" leads to the description of

"Electron is a $\underbrace{\text{spin-}1/2}_{\text{magnitude } |S| = \sqrt{\frac{3}{4}}\hbar \text{ actually}}$ (spin-half) particle"

[+] Because Spin AM is very important, it has its own notation "S" instead of the general "J", its own quantum numbers s, m_s instead of j, m_j

Important Remarks

- Electron's Spin is a property of electron

  [In contrast, electron's orbital AM is NOT an intrinsic property.

  For electron in 1s (ψ_{100}), its $|L| = 0$, $L_z = 0$

  For electron in 2p (say ψ_{210}), its $|L| = \sqrt{2}\hbar$, $L_z = 0$

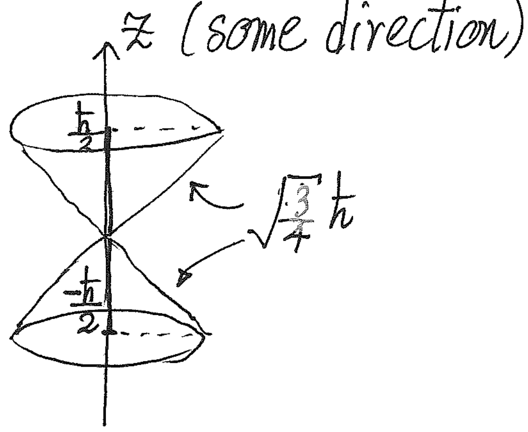
  $\therefore$ Orbital AM depends on the atomic state it is in.]

- Spin AM eigenstates cannot be expressed as a wavefunction of space (x, y, z)

  [Electron's Spin is unrelated to electron's prob. density and thus $\psi(x, y, z)$, because it is intrinsic]

- A spinning electron picture is just a model

- Can also use a vector model as a picture of spin



- Don't mistaken that it is the electron orbiting (No! No! No!)

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spin is a vector
(precession)

It is just a model.

The results come from the <u>commutators</u>.

- Existence of spin AM is something beyond the Schrödinger Equation [does not come from solving TISE for atoms]

- Dirac (1927): Dirac's theory of an electron

  $\underbrace{\text{Dirac Equation for an electron}}_{\text{gives electron's spin in its solutions}}$ [<u>relativistic</u> quantum mechanics]

  $\therefore$ $\boxed{\text{Spin angular Momentum is a relativistic effect}}$

No wonder $\underbrace{\underline{\text{Schrödinger Equation}}}$ could not give it

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(\hat{x}) \qquad \frac{p^2}{2m} = \frac{1}{2}mv^2 \longleftarrow \text{Non-relativistic starting point}$$

- Since exp'tal results are consistent with general QM Angular momentum results, the spin AM operators must follow

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z \ , \ [\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x \ , \ [\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$$

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$$

$$[\hat{S}^2, \hat{S}_x] = [\hat{S}^2, \hat{S}_y] = [\hat{S}^2, \hat{S}_z] = 0$$

with exp'tal results indicating $s = 1/2$ only for electrons.

- Using $|s, m_s\rangle$ for $|j, m_j\rangle$ for $\hat{S}^2$ and $\hat{S}_z$

  Electron Spin only allows

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle \quad \text{and} \quad |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \qquad (11) \qquad \text{(formally)}$$

  always $s=\frac{1}{2}$ always $s=\frac{1}{2}$

- Boring to keep on writing the first "½" (always "½"),

  <u>short hand</u>: $|m_s = \tfrac{1}{2}\rangle$, $|m_s = -\tfrac{1}{2}\rangle$ (12)

  OR $|\tfrac{1}{2}\rangle$, $|-\tfrac{1}{2}\rangle$ (13)

  OR $|\uparrow\rangle$, $|\downarrow\rangle$ (14) (Quantum Information)

  OR α , β (15) (Quantum Chemistry)

E. Matrix Representation of Spin Angular Momentum

- Simple for spin ($\because$ $s = 1/2$, $m_s = 1/2, -1/2$ only two values)
- Arbitrarily, choose a matrix representation that the matrix for S_z is diagonal

[S_z has two eigenvalues: $\frac{\hbar}{2}$, $-\frac{\hbar}{2}$
[S_z is diagonal

(16) $$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\nearrow$ eigenvalue $\frac{\hbar}{2}$ $\leftrightarrow$ eigenvector (eigenstate) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\searrow$ eigenvalue $-\frac{\hbar}{2}$ $\leftrightarrow$ eigenvector (eigenstate) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore$ $|1/2, 1/2\rangle = |m_s = 1/2\rangle = |1/2\rangle = |\uparrow\rangle = \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in this representation

$|1/2, -1/2\rangle = |m_s = -1/2\rangle = |-1/2\rangle = |\downarrow\rangle = \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in this representation

How about $[S_x]$, $[S_y]$ matrices?

- Conditions: $[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$, $[\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x$, $[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$

$$\underbrace{\hat{S}^2}_{2\times 2} = \underbrace{\hat{S}_x^2}_{2\times 2} + \underbrace{\hat{S}_y^2}_{2\times 2} + \underbrace{\hat{S}_z^2}_{2\times 2} \quad \text{has only } \tfrac{3}{4}\hbar^2 \text{ as eigenvalue}$$

(17) $$\hat{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{two eigenvalues } \left(\tfrac{3}{4}\hbar^2 \text{ and } \tfrac{3}{4}\hbar^2\right)$$

(18) $$\boxed{\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{work!}}$$

(16) $$\boxed{\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Eqs. (16), (18) give the most economical way of representing $s = 1/2$ angular momentum.

Ex: Check all commutation relations

Ex: Eigenvalues of $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

[Meaning: Measure any component (along any direction, not only x, y, z directions), only two possible outcomes of $+\frac{\hbar}{2}$, $-\frac{\hbar}{2}$]

- Eigenstate of $\hat{S}_z$ ($\hat{S}_x$) is NOT an eigenstate of $\left\{\begin{matrix}\hat{S}_x \\ \hat{S}_y\end{matrix}\right\}$ $\left(\left\{\begin{matrix}\hat{S}_y \\ \hat{S}_z\end{matrix}\right\}\right)$

Pauli Matrices

$$\hat{S}_x = \frac{\hbar}{2}[\sigma_x] \; ; \; \hat{S}_y = \frac{\hbar}{2}[\sigma_y] \; ; \; \hat{S}_z = \frac{\hbar}{2}[\sigma_z] \qquad (19)$$

$$\underbrace{[\sigma_x] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; [\sigma_y] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; [\sigma_z] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{Pauli}^{\dagger}\text{ spin matrices (or Pauli matrices)}} \qquad (20)$$

- $[\sigma]$'s properties are keys to spin physics
- $[\sigma]$'s are important in relativistic QM

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† Pauli was awarded the 1945 Nobel Physics Prize "for the discovery of the Exclusion Principle, also called the Pauli Principle". He did much more than that!